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## Canonical Discretization

### I. Discrete faces of (an)harmonic oscillator

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#### Abstract

A certain notion of canonical equivalence in quantum mechanics is proposed. It is used to relate quantal systems with discrete ones. Discrete systems canonically equivalent to the celebrated harmonic oscillator as well as the quartic and the quasi-exactly-solvable anharmonic oscillators are found. They can be viewed as a translation-covariant discretization of the (an)harmonic oscillator preserving isospectrality. The notion of the  $q$ -deformation of the canonical equivalence leading to a dilatation-covariant discretization preserving polynomiality of eigenfunctions is also presented.

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Two classical mechanical systems related by a canonical transformation are equivalent (for general discussion see, for example, [1]). It is quite natural to try to adopt a similar notion for the quantum mechanical systems making a difference that classical canonical transformations should be replaced by their quantum counterpart, which means that the Poisson bracket is replaced to the Lie bracket,  $[P(p, q), Q(p, q)] = 1$ . Present work is an attempt to explore this natural definition taking as an example both (an)harmonic oscillator and a certain particular type of quantum canonical transformations. There are well-known difficulties on this way and we will try to indicate and overtake them. Meanwhile, we will call systems related through a quantum canonical transformation *canonically equivalent*.

Let us remind that any quantum-mechanical system with a Hamiltonian  $\mathcal{H}(x, \hat{p})$  is intrinsically related to a Heisenberg algebra  $[x, \hat{p}] = -i$ . Hamiltonian can be treated as an operator (or, an element of the Heisenberg-Weyl algebra) acting in the quantum-mechanical phase space which is defined as an object consisting of the universal enveloping Heisenberg algebra (the Heisenberg-Weyl algebra) and the vacuum  $|0\rangle$ , such that  $\hat{p}|0\rangle = 0$ . From this point of view, canonical transformations are nothing but changes of variables in the phase space preserving polarization or, in other words, preserving the property that the Heisenberg algebra is the underlying algebra of the system. This construction can be generalized by considering formally the Hamiltonian as an operator acting in the  $q$ -deformed phase space. It implies the  $q$ -deformed Heisenberg algebra as the underlying algebra,  $x\hat{p} - q\hat{p}x = -i$  of the system<sup>1</sup>, and taking a class of transformations preserving this algebraic structure. It leads naturally to the Fock space formalism as an adequate language of our study.

The harmonic oscillator as well as anharmonic, both classical and quantum plays a fundamental role in physics. A particular goal of present study is to find discrete systems those can be related to quantum-mechanical (an)harmonic oscillator via a quantum canonical transformation. It will be considered two types of discrete systems: translation-covariant (uniform grid) (i) and dilatation-covariant (exponential grid) (ii). In what follows it will be used a Fock space formalism. In the next paper the case of Coulomb, Poschlö-Teller and Morse potentials will be presented [2].

Important general remark should be made in row. Generically, if two systems I and II are related by a gauge transformation and/or by

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<sup>1</sup>It also allows a possible modification of the Hamiltonian by inserting the parameter  $q$  in appropriate places

a change of variables as equivalent, namely,

$$\mathcal{H}_I(x) = g(y)^{(-1)} \mathcal{H}_{II}(y) g(y) |_{y=y(x)}$$

these systems will be treated as equivalent.

### 1. Harmonic oscillator in Fock space

The Hamiltonian of harmonic oscillator is defined by

$$\mathcal{H} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\omega^2}{2} x^2, \quad (1)$$

where  $\omega$  is the oscillator frequency. The configuration space of (1) is the whole real line,  $x \in (-\infty, \infty)$ . The eigenfunctions and eigenvalues are given by

$$\Psi_k(x) = H_k(\sqrt{\omega}x) e^{-\omega \frac{x^2}{2}}, \quad E_k = \omega(k + 1/2), \quad k = 0, 1, \dots \quad (2)$$

where  $H_k(y)$  is the  $k$ th Hermite polynomial in standard notation. Without a loss of generality we put all normalization constants in (2) equal to 1. The Hamiltonian (1) is  $Z_2$ -invariant,  $x \rightarrow -x$ , which leads to two families of eigenstates: even and odd, or, in other words, symmetric and anti-symmetric with respect of reflection, correspondingly. This property is coded in parity of the Hermite polynomials and is revealed by the relation:

$$H_{2n+p}(\sqrt{\omega}x) = \omega^{\frac{p}{2}} x^p L_n^{(p-\frac{1}{2})}(\omega x^2), \quad n = 0, 1, \dots \quad (3)$$

where  $L_n^{(\alpha)}(y)$  is the  $n$ th associated Laguerre polynomial in standard notation, and the parameter  $p = 0, 1$  determines parity,  $P = (-1)^p$ . Hereafter we can call

$$\Psi_0^{(p)}(x) = x^p e^{-\omega \frac{x^2}{2}}, \quad (4)$$

the ground state (the lowest energy state) of parity  $P$ . Thus, the formula (4) makes an unification of both possible values of parity and for the sake of simplicity we will call (4) the ground state eigenfunction without specifying parity.

Let us make a gauge rotation of the Hamiltonian (1) taking as a gauge factor the ground state eigenfunction (4) and changing the variable  $x$  to  $y = \omega x^2$ , which incorporate the reflection symmetry. After dropping out non-essential constant term,  $(\frac{1}{2} + p)$ , which cause a change the reference point for energy only, we get an operator

$$h(y, \partial_y) = \frac{1}{\omega} (\Psi_0^{(p)}(x))^{-1} \mathcal{H} \Psi_0^{(p)}(x) |_{y=\omega x^2} = -2y \partial_y^2 + 2(y - p - \frac{1}{2}) \partial_y \quad (5)$$

with the spectrum  $E_n = 2n$ , where  $n = 0, 1, 2, \dots$ <sup>2</sup>. Now the eigenvalue problem for the operator (5) is defined on the half-line  $y \in [0, \infty)$ . The operator (5) simultaneously describes a family of eigenstates of positive parity if  $p = 0$  and a family of eigenstates of negative parity if  $p = 1$ . We will call (5) *the algebraic form* of the Hamiltonian of the harmonic oscillator. The word ‘algebraic’ reflects the fact that the operator (5) has a form of linear differential operator with polynomial coefficients and furthermore possesses infinitely-many polynomial eigenfunctions. The latter implies that any eigenfunction can be found by algebraic means by solving a system of linear algebraic equations.

The algebraic form (5) admits a generalization of the original Hamiltonian (1) we started with. If we assume that the parameter  $p$  can take any real value,  $p > -1/2$ , one can make an inverse gauge transformation of the operator (5) back to the Hamiltonian form and we arrive at

$$\begin{aligned} & \omega y^{p/2} e^{-y/2} \left[ -2y \partial_y^2 + 2(y - p - \frac{1}{2}) \partial_y \right] y^{-p/2} e^{y/2} \Big|_{x=\sqrt{\frac{y}{\omega}}} \\ &= \left[ -\frac{1}{2} \partial_x^2 + \frac{\omega^2}{2} x^2 + \frac{p(p-1)}{2x^2} \right] \equiv \mathcal{H}_k, \end{aligned} \quad (6)$$

which is known in literature as Kratzer Hamiltonian. It is worth to mention that this Hamiltonian coincides with 2-body Calogero Hamiltonian. Also this Hamiltonian appears as a Hamiltonian of the radial motion of multidimensional spherical-symmetrical harmonic oscillator. Hereafter we will continue to call the system characterized by the Hamiltonian (6) *the harmonic oscillator*.

The resulting Hamiltonian (6) is characterized by the eigenfunctions

$$\Psi_k(x) = x^p L_n^{(p-\frac{1}{2})}(\omega x^2) e^{-\omega \frac{x^2}{2}}, \quad (7)$$

which coincides with (2) at  $p = 0, 1$ . The spectrum (6) is still equidistant with energy gap  $\omega$  and after appropriate shift of the reference point it coincides with the spectrum of the original harmonic oscillator (1). Thus, the deformation of (1) to (6) is isospectral which is, of course, well-known.

In order to move ahead let us introduce a notion of the Fock space. It will be a natural formalism to study canonical transformations. Take two operators  $a$  and  $b$  obeying the commutation relation

$$[a, b] \equiv ab - ba = I, \quad (8)$$

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<sup>2</sup>It is necessary to emphasize that the spectrum is defined by action of the operator  $2y\partial_y$  in (5) on monomial:  $(2y\partial_y)y^n = 2ny^n$ . It is the only operator of zero-grading in (5)

with the identity operator  $I$  on the r.h.s. – they span a three-dimensional Lie algebra which is called the Heisenberg algebra  $h_3$ . By definition the universal enveloping algebra of  $h_3$  is the algebra of all normal-ordered polynomials in  $a, b$ : any monomial is taken to be of the form  $b^k a^m$ <sup>3</sup>. If, besides the polynomials, all entire functions in  $a, b$  are considered, then the *extended* universal enveloping algebra of the Heisenberg algebra appears or in other words, the extended Heisenberg-Weyl algebra. In the (extended) Heisenberg-Weyl algebra one can find the non-trivial embedding of the Heisenberg algebra: non-trivial elements obeying the commutation relations (8), whose can be treated as a certain type of quantum canonical transformations. We say that the (*extended*) *Fock space*,  $\mathcal{F}$  is determined if we take the (extended) universal enveloping algebra of the Heisenberg algebra and attach to it the vacuum state  $|0\rangle$  defined as

$$a|0\rangle = 0. \quad (9)$$

It is easy to check that the following statement holds. If the operators  $a, b$  obey the commutation relation (8), then the operators

$$\begin{aligned} J_n^+ &= b^2 a - n b, \\ J_n^0 &= b a - \frac{n}{2}, \\ J_n^- &= a, \end{aligned} \quad (10)$$

span the  $sl_2$ -algebra with the commutation relations:

$$[J^0, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = -2J^0,$$

where  $n \in \mathbf{C}$ <sup>4</sup>. For the realization (10) the quadratic Casimir operator is equal to

$$C_2 \equiv \frac{1}{2}\{J_n^+, J_n^-\} - J_n^0 J_n^0 = -\frac{n}{2}\left(\frac{n}{2} + 1\right), \quad (11)$$

where  $\{, \}$  denotes the anticommutator and is  $c$ -number. If  $n \in \mathbb{Z}_+$ , then (10) possesses a finite-dimensional, irreducible representation in Fock space leaving invariant the linear space of polynomials in  $b$  acting on vacuum:

$$\mathcal{P}_n(b) = \langle 1, b, b^2, \dots, b^n | 0 \rangle, \quad (12)$$

of dimension  $\dim \mathcal{P}_n = (n+1)$ . It is evident that any operator which is a polynomial in generators  $J_n^{+,0,-}$  preserves the space  $\mathcal{P}_n(b)$  (and converse is also correct [3]). Such an operator we call  *$sl_2$ -quasi-exactly-solvable*

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<sup>3</sup>Sometimes this is called the Heisenberg-Weyl algebra

<sup>4</sup>For details and discussion see, for example, [4]

operator. Thus, the most general  $sl_2$ -quasi-exactly-solvable operator in the Fock space having a form of a polynomial in  $a$  of degree not higher than two is given by

$$T_2 = c_{++}J_n^+J_n^+ + c_{+0}J_n^+J_n^0 + c_{00}J_n^0J_n^0 + c_{0-}J_n^0J_n^- + c_{--}J_n^-J_n^- + c_+J_n^+ + c_0J_n^0 + c_-J_n^- + c, \quad (13)$$

where  $c_{\alpha\beta}, c_\alpha, c$  are arbitrary c-numbers, or after the substitution (10) in explicit form

$$T_2(b, a) = -P_4(b)a^2 + P_3(b)a + P_2(b), \quad (14)$$

where the  $P_j(b)$  are polynomials of  $j$ th order with coefficients related to  $c_{\alpha\beta}, c_\alpha, c$  and  $n$ .

The spaces  $\mathcal{P}_n$  possess a property that  $\mathcal{P}_n \subset \mathcal{P}_{n+1}$  for each  $n \in \mathbb{Z}_+$  and form an infinite flag (filtration) and

$$\bigcup_{n \in \mathbb{Z}_+} \mathcal{P}_n = \mathcal{P}.$$

Hence it is evident that any operator which is a polynomial in generators  $J_n^{0,-}$  only preserves the flag of  $\mathcal{P}$  [3]. Such an operator we call  *$sl_2$ -exactly-solvable operator*. Thus, the most general  $sl_2$ -exactly-solvable operator in the Fock space having a form of a polynomial in  $a$  of degree not higher than two is given by

$$E_2 = c_{00}J^0J^0 + c_{0-}J^0J^- + c_{--}J^-J^- + c_0J^0 + c_-J^- + c, \quad (15)$$

where  $J^{\pm,0} \equiv J_0^{\pm,0}$  and  $c_{\alpha\beta}, c_\alpha, c$  are arbitrary c-numbers. After the substitution (10) in explicit form

$$E_2(b, a) = -Q_2(b)a^2 + Q_1(b)a + Q_0, \quad (16)$$

where the  $Q_j(b)$  are polynomials of  $j$ th order with coefficients related to  $c_{\alpha\beta}, c_\alpha, c$ .

Now we can introduce a notion of the spectral problem in the Fock space. Let  $L(b, a)$  is an element of the Heisenberg-Weyl algebra. By definition, to solve the spectral problem for the operator  $L(b, a)$  is to find a set of the elements  $\{\phi(b)\}$  in the Heisenberg-Weyl algebra and a corresponding set of parameters  $\{\lambda\}$  for those the equation

$$L(b, a)\phi(b)|0\rangle = \lambda\phi(b)|0\rangle \quad (*)$$

is fulfilled. We will call  $\{\phi(b)\}$  and  $\{\lambda\}$  the eigenfunctions and eigenvalues, respectively. Attempting to study the spectral problem  $(*)$  immediately leads to a delicate question about convergence of the operator series. In order to avoid possible difficulties we will restrict our

consideration by the cases when the polynomial in  $b$  eigenfunctions appear.

## 2. Translation-covariant discretization

Take as an example the  $sl_2$ -exactly-solvable operator of the following form

$$h^{(1)}(b, a) = -2J^0 J^- + 2J^0 - 2(p + \frac{1}{2})J^- = -2ba^2 + 2(b - p - \frac{1}{2})a, \quad (17)$$

where  $p$  is a parameter and  $J^{\pm,0} \equiv J_0^{\pm,0}$  (see (10)). Simple analysis leads to a statement that the eigenfunctions of  $h_f$  are the associated Laguerre polynomials of the argument  $b$ ,  $L_n^{(p-\frac{1}{2})}(b)$  and their eigenvalues,  $E_n = 2n$ ,  $n = 0, 1, 2, \dots$ <sup>5</sup>.

As the next step we consider two different realization of the Heisenberg algebra (8) in terms of differential and finite-difference operators. A traditional realization of (8) appearing in text-books is the so-called coordinate-momentum representation:

$$a = \frac{d}{dy} \equiv \partial_y, \quad b = y, \quad (18)$$

(see, for example, the book by Landau and Lifschitz [5]), where the operator  $b = y$  stands for the multiplication operator:  $bf(y) = yf(y)$ . In this case the vacuum is a constant and without a loss of generality we put  $|0\rangle = 1$ . However, there exists another realization of (8) in terms of finite-difference operators. It is a finite-difference analogue of (18):

$$a = \mathcal{D}_+, \quad b = (y + \alpha)(1 - \delta\mathcal{D}_-), \quad (19)$$

where

$$\mathcal{D}_{\pm}f(y) = \frac{f(y \pm \delta) - f(y)}{\pm\delta},$$

(cf.[6]) is the finite-difference operator. It represents what can be called a  $\delta$ -discretization of the derivative:  $\mathcal{D}_{\pm} \rightarrow \partial_y$ , if  $\delta \rightarrow 0$  and, thus, which can be called  $\delta$ -derivative<sup>6</sup>. In general,  $\delta, \alpha$  can be any complex numbers and  $\mathcal{D}_{\pm}(-\delta) = \mathcal{D}_{\mp}(\delta)$ . It is necessary to emphasize that from physical point of view the operator  $b$  in (19) is nothing but

<sup>5</sup>As in (5) the eigenvalues are defined by action of the operator  $(2ba)$  in (13) on monomials:  $(2ba)b^n = 2nb^n + 2b^{n+1}a$ . The second term disappears after action on the vacuum

<sup>6</sup>This finite-difference operator is also known in mathematics literature as the Nörlund derivative (see, for example,[7]) while we prefer to use a name  $\delta$ -derivative in order to distinguish from  $q$ -derivative (see below) – another type of discretization

the canonical-conjugate to a discrete momentum operator defined by  $\delta$ -derivative. So, (19) represents two-parametric family of quantum canonical transformations. For  $\alpha = 0$  and  $\delta \rightarrow 0$ , the formulas (19) become (18). It allows to interpret (19) as the continuous deformation of (18).

A remarkable property of this realization is that the vacuum can be taken as a constant <sup>7</sup> and thus, without loss of generality, it can be placed as  $|0\rangle = 1$  for both cases (18)-(19). Realization (19) is translation-covariant: under a linear shift of variable  $y \rightarrow y + B$  the functional form of (19) is preserved.

Substitution of (18) into (17) leads to the operator (5) – the algebraic form of the Hamiltonian of the harmonic oscillator. Thus, the operator (17) can be called the *the Hamiltonian of the harmonic oscillator in the Fock space*. It is evident that the procedure of realization of the Heisenberg generators  $a, b$  by concrete operators (differential, finite-difference, discrete) provided that the vacuum remains unchanged leaves any polynomial operator in  $a, b$  isospectral.

Now let us study another ‘face’ of harmonic oscillator by substituting the realization of the Heisenberg algebra by finite-difference operators (19) into (17). It results to

$$h_\delta^{(1)}(y, D_\pm) = -\frac{2}{\delta}[y + \alpha + \delta(p + \frac{1}{2})]D_+ + 2(1 + \frac{1}{\delta})(y + \alpha)D_- . \quad (20)$$

In this realization the corresponding spectral problem (\*) can be written as

$$\begin{aligned} & -\frac{2}{\delta^2}[y + \alpha + \delta(p + \frac{1}{2})]\phi(y + \delta) + \frac{2}{\delta}[(1 + \frac{2}{\delta})y + \alpha + p + \frac{1}{2}]\phi(y) \\ & - \frac{2}{\delta}(1 + \frac{1}{\delta})(y + \alpha)\phi(y - \delta) = E\phi(y) . \end{aligned} \quad (21)$$

It is worth to note that the Askey condition that the sum of the coefficients in front of unknown functions should be zero is fulfilled at  $E = 0$ . It is equivalent to a statement that the equation (21) possess an eigenfunction which is a constant.

In general, the operator  $h_\delta(y, D_\pm)$  is a non-local, three-point, finite-difference and translation-covariant operator. This operator is canonically-equivalent to the harmonic oscillator. Pictorially it is illustrated by Fig.1.

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<sup>7</sup>Any  $\delta$ -periodic function can be chosen as a vacuum for (19) as well



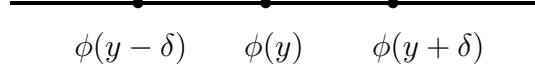


Fig. 1. Graphical representation of the operator (20)

So, in the  $y$ -space we have the uniform grid, linear lattice. However, from the point of view of the original configuration space,  $x$ -space where the harmonic oscillator (1) is defined we have a square lattice.

In order to find the eigenfunctions of (20) a certain trick can be used. One can easily show that the following equality holds

$$[(y + \alpha)e^{-\delta\partial_y}]^n I = (y + \alpha)^{(n)} I ,$$

where  $(y + \alpha)^{(n+1)} = (y + \alpha)(y + \alpha - \delta) \dots (y + \alpha - n\delta)$  is a so-called *quasi-monomial* or *generalized monomial* and  $I$  is the identity operator. Then making use this relation it is not difficult to prove that the eigenfunctions of (20) remain polynomials in  $y$  and, furthermore, they can be found explicitly in rather elegant way

$$\phi_n(y) = \hat{L}_n^{(p-\frac{1}{2})}(y, \delta) = \sum_{\ell=0}^n a_\ell^{(p-\frac{1}{2})} (y + \alpha)^{(\ell)} , \quad (22)$$

where  $a_\ell^{(p-\frac{1}{2})}$  are the coefficients in the expansion of the Laguerre polynomials,  $L_n^{(p-\frac{1}{2})}(y) = \sum_{\ell=0}^n a_\ell^{(p-\frac{1}{2})} y^\ell$ . We call these polynomials the *modified associated Laguerre polynomials*. Simultaneously, the eigenvalues in the equation (21) remain equal to  $(2n), n = 0, 1, 2 \dots$  and they are the same as the eigenvalues of the harmonic oscillator problem (1) as well as (6) and (13). Thus, one can say that the operator (20) defines a *finite-difference or  $\delta$ -discrete algebraic form of harmonic oscillator* Hamiltonian. Without loss of generality one can place  $\alpha = 0$  in (20), (21), (22).

There exists two non-trivial particular cases of (20). The first case corresponds to the spacing  $\delta = -1$ . It leads to a disappearance of the term proportional to  $D_-$  and, thus, the operator (20) becomes two-point operator (!)

$$h_\delta^{(1)}(y, D_\pm) = 2[y + \alpha - (p + \frac{1}{2})]D_+|_{\delta=-1} \equiv 2[y + \alpha - (p + \frac{1}{2})]D_-|_{\delta=1} . \quad (23)$$

It is worth noting that the spectrum in this case is defined by the term  $(2yD_+|_{\delta=-1})$  stemming from the negative grading term  $(2ba^2)$  in (17). Breaking a condition of performing the canonical transformations

we re-insert the parameter  $\delta$  in (23) in naive, straightforward manner:  $D_+|_{\delta=-1} \rightarrow D_+(\delta)$ . It results to the operator

$$h_\delta^{(1)}(y, D_\pm) = 2[y + \alpha - (p + \frac{1}{2})]D_+ , \quad (24)$$

which is isospectral to (5), (17) for any  $\delta$ . In limit  $\delta \rightarrow 0$  it leads to the first-order differential operator

$$h_\delta^{(1)}(y, \partial_y) = 2[y + \alpha - (p + \frac{1}{2})]\partial_y . \quad (25)$$

Although, the operators (23), (24) are isospectral to various forms of the harmonic oscillator (1), (6), those operators are not related to each other by a quantum canonical transformation and therefore are not canonically equivalent. Above-mentioned isospectral transition from the second-order differential operator (5) to the first order differential operator (25) or from finite-difference one (20) to (24) reminds celebrated Bargmann transformation (see for example [8]). It is necessary to emphasized that the operator (24) has infinitely-many polynomial eigenfunctions, however, unlike the operators (13),(16), the operator (20) is not  $sl_2$ -exactly-solvable (!). Also it can not be rewritten in terms of the generators  $a, b$  of the Heisenberg algebra (8) and does not belong to the Fock space (see discussion below).

Another important particular case occurs if the spacing  $\delta = -2$ . In this case the operator (20) again becomes two-point one

$$-\frac{1}{2}[y + \alpha - 2(p + \frac{1}{2})]\phi(y-2) + \frac{1}{2}(y + \alpha)\phi(y+2) = \tilde{E}\phi(y) , \quad (26)$$

where a new spectral parameter  $\tilde{E} = E + \alpha + p + \frac{1}{2}$ .

*Remark.* The function  $\phi(y)$  in the r.h.s. of (21) can be replaced by  $\phi(y+\delta)$  or  $\phi(y-\delta)$ , or by a linear combination of  $\phi(y\pm\delta), \phi(y)$ . It does not change the statement that (21) has infinitely-many polynomial eigenfunctions. This procedure preserves isospectral property. However, such changes of the r.h.s. lead to a replacement of the original standard spectral problem (\*) by a ‘generalized’ spectral problem. In this case the r.h.s. contains an operator other than the operator of multiplication on a function. In general, a physical relevance of such right-hand-sides is unclear. It is worth to note that it reminds the Sturm representation of the Coulomb problem when the energy is kept fixed but a set of discrete electric charges for which a corresponding excited state energy is equal to this energy is studied. In this formulation the r.h.s.

of the Coulomb problem contains the Coulombic interaction potential as the weight factor.

A natural question can be posed about the most general second-order linear differential operator, which (i) is isospectral to the harmonic oscillator (1),(6); (ii) has infinitely-many polynomial eigenfunctions and (iii) related to (5) by a canonical transformation. Following the Theorem [3] one can derive the most general operator with above properties

$$h_{\delta,g}(y, \partial_y) = -2(AJ^0 + BJ^-)J^- + 2J^0 - 2CJ^- = \\ -2(Ay + B)\partial_y^2 + 2(y - C)\partial_y \quad (27)$$

where  $A, B, C$  are arbitrary constants and the generators (10) are realized by differential operators (18). However, by a linear change of variable,  $y \rightarrow \alpha y + \beta$  with appropriate  $\alpha \neq 0, \beta$  the operator (27) can be transformed into (5). It reflects a fact that the linear space of polynomials  $\mathcal{P}_n(x)$  is invariant under a linear transformation:  $x \rightarrow \gamma_1 x + \gamma_2, \gamma_1 \neq 0$ . Thus, without loss of generality we can put  $A = \alpha = 1$  and also  $C = p + 1/2$ . The eigenfunctions of (27) remain the Laguerre polynomials but of a shifted argument,  $L_n^{(p-\frac{1}{2})}(y + \beta)$ . It leads to a statement that among the second-order differential operators there exists no non-trivial isospectral deformation of the harmonic oscillator potential preserving polynomiality of the eigenfunctions.

The operator (27) can be rewritten in the Fock space formalism by using (18)

$$h_g(b, a) = -2(b + B)a^2 + 2(b - C)a. \quad (28)$$

It is evident that the operator (28) is the most general second order polynomial in  $a$ , which is isospectral to (17) and also preserves the space of polynomials (12). By substitution (19) into the operator (28) it becomes a finite-difference operator

$$h_{\delta,g}(y, D_{\pm}) = -2BD_+^2 - \frac{2}{\delta}[y + \alpha + \delta C]D_+ + 2(1 + \frac{1}{\delta})(y + \alpha)D_- \quad (29)$$

(cf. (20)). Thus, in the realization by finite-difference operators the corresponding spectral problem (\*) has the form

$$\begin{aligned} & \frac{-2B}{\delta^2}\phi(y + 2\delta) - \frac{2}{\delta^2}[y + \alpha - 2B + \delta C]\phi(y + \delta) \\ & + \frac{2}{\delta}[(1 + \frac{2}{\delta})(y + \alpha) - \frac{B}{\delta} + C]\phi(y) - \frac{2}{\delta}(1 + \frac{1}{\delta})(y + \alpha)\phi(y - \delta) = E\phi(y), \end{aligned} \quad (30)$$

and is characterized by existence of infinitely family of eigenfunctions given by polynomials in  $y$ .

The operator  $h_\delta(y, D_\pm)$  now becomes the four-point finite-difference operator, see Fig.2 .

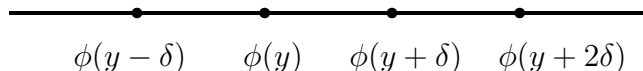


Fig. 2. Graphical representation of the operator (29).

It is quite surprising that a simple transformation like linear shift of variable  $y$  in differential operator (20) (which leads to nothing non-trivial, see above) leads to occurrence of the extra point in the isospectral finite-difference counterpart changing a type of its non-locality. Nevertheless, these operators remain to be canonically equivalent being related each other by a canonical transformation. So, by a canonical transformation one can change a non-local nature of finite-difference operators.

It is quite interesting to abandon the condition of canonical equivalence and pose a question about the most general differential (finite-difference) operators isospectral to the harmonic oscillator and possessing the infinitely-many polynomial eigenfunctions. One can easily show that ignoring the condition of canonical equivalence leads to nothing new for differential operators. However, for the case of finite-difference operators much wider class of operators occurs than (20) or (29). As a natural constraint we have to impose a condition of maximal number of points in the finite-difference operators we search for. Let us consider three-point operators (see Fig.1). A simple analysis leads to a statement that the three-parametric operator

$$\tilde{h}_\delta(y, D_\pm) = 2[(1 - A)y + C_+]D_+ + 2[Ay + C_-]D_- , \quad (31)$$

where  $A, C_\pm$  are parameters, is the most general three-point finite-difference operator with infinitely-many polynomial eigenfunctions, which is isospectral to the harmonic oscillator. Of course, the operators (16) and (18) are particular cases of (31). The form (31) can be understood as a consequence of the Theorem which states that *any finite-difference operator  $h_\delta(y, D_\pm)$  preserving the infinite flag of polynomials*

$$\mathcal{P}_n(y) = \langle 1, y, y^2, \dots, y^n \rangle , \quad (32)$$

*should have a representation in terms of the operators:*

$$J_\pm^- = D_\pm , \quad J_\pm^0 = yD_\pm . \quad (33)$$

As a remark we should note that the generators (33) do not span an algebra closed with respect to commutators. However, if we consider a linear space spanned by (33), then the Heisenberg algebra (18) appears as its subspace.

It is worth to mention that the canonical discretization of the harmonic oscillator can be made directly in the configuration  $x$ -space where the original harmonic oscillator (1) is defined. If we make a gauge transformation (5) with (4) at  $p = 0$  as the gauge factor but without the change of variable, then

$$h^{(2)}(x, \partial_x) = \frac{1}{\omega}(\Psi_0^{(0)}(x))^{-1} \mathcal{H} \Psi_0^{(0)}(x) = -\frac{1}{2\omega} \partial_x^2 + x \partial_x, \quad (34)$$

is another algebraic form of the harmonic oscillator (cf.(5)). The operator (34) has the Hermite polynomials,  $H_k(\sqrt{\omega}x)$ , as the eigenfunctions. Now we can make  $\delta$ -discretization firstly rewriting (34) in the Fock space formalism

$$h^{(2)}(b, a) = -\frac{1}{2\omega} a^2 + ba = -\frac{1}{2\omega} J^- J^- + J^0, \quad (35)$$

(cf.(17)) and then realizing  $a, b$  by finite-difference operators (19)

$$h_\delta^{(2)}(x, D_\pm) = -\frac{1}{2\omega} D_+^2 + (x + \alpha) D_- . \quad (36)$$

This operator is canonically-equivalent to the harmonic oscillator, it is defined on the uniform linear lattice in  $x$ -space and has the eigenvalues  $E_k = k$ ,  $k = 0, 1, 2, \dots$ . However, it does not possess the symmetry  $x \rightarrow -x$  unlike the  $\delta$ -discretized, canonically-equivalent operator (20) in  $y$ -space. The operator (36) is non-local, four-point operator with eigenfunctions

$$\phi_k(x) = \sum_{\ell=0}^k a_\ell \omega^{\frac{\ell}{2}} (x + \alpha)^{(\ell)} . \quad (37)$$

where  $z^{(\ell)}$  is quasi-monomial and  $a_\ell$  are the coefficients in the expansion of the Hermite polynomials,  $H_k(x) = \sum_{\ell=0}^k a_\ell x^\ell$ . These eigenfunctions are closely related to the polynomials which can be called the *modified Hermite polynomials*

$$\hat{H}_k(x, \delta) = \sum_{\ell=0}^k a_\ell x^{(\ell)}, \quad (38)$$

and, thus,

$$\phi_k(x) = \hat{H}_k(\sqrt{\omega}(x + \alpha), \sqrt{\omega}\delta) .$$

Summarizing, in this Section we presented two  $\delta$ -discretized operators which are canonically-equivalent to the harmonic oscillator, (20) and (36), which are defined on the uniform grid in  $y$ - and  $x$ -spaces, respectively. In spite of the fact that they are represented by different elements of the Heisenberg-Weyl algebra, they have infinitely-many polynomial eigenfunctions and are isospectral to the original harmonic oscillator (1).

### 3. Dilatation-covariant discretization

Apart from the translation-covariant discretization given by  $\delta$ -derivative (see (19)) there exists a dilatation-covariant discretization based on  $q$ -derivative,  $\mathcal{D}_q$ , which is also called the Jackson symbol

$$\mathcal{D}_q f(y) = \frac{f(qy) - f(y)}{y(1 - q)} ,$$

where  $q$  is a complex number. This Section will be devoted to a brief discussion of the dilatation-covariant discretization or  $q$ -discretization. First of all by following the above-mentioned philosophy a natural question can be posed about an existence of a quantum canonical conjugate to  $q$ -derivative – the operator which obeys together with  $\mathcal{D}_q$  the commutation relations (8). Up to our knowledge a definite answer is not found so far and very likely such an object does not exist in terms of well-defined operators. However, it is well known that the derivative  $\mathcal{D}_q$  appears naturally in connection to a realization of quantum algebras in action on functions in one and several variables. Thus, it looks reasonable to explore a generalization of the underlying Heisenberg algebra described above to the case of the  $q$ -deformed (quantum) Heisenberg algebra. Hence, we will study ‘deformed’ quantum systems possessing a  $q$ -deformed (quantum) Heisenberg algebra as a hidden algebra instead of the standard Heisenberg algebra (8) (see discussion in Introduction).

In order to proceed let us ask first what would happen if in the expressions (17), (28) the operators  $a, b$  are not the generators of the Heisenberg algebra (8) but the generators of the  $q$ -deformed Heisenberg algebra

$$[a, b]_q \equiv ab - qba = 1 , \quad (39)$$

where  $q$  is a parameter. Following the Theorem proved in [3], one can demonstrate that within the  $q$ -deformed Fock space built on the  $q$ -deformed Heisenberg algebra (39) there exists the flag  $\mathcal{P}$  of linear spaces of polynomials in  $b$  (see (12)), which is preserved by the operators (17), (28). By a simple calculation one can find the eigenvalues of

the operators (17), (28) in the spectral problem (\*)

$$E_n^{(q)} = 2\{n\} , \quad n = 0, 1, \dots , \quad (40)$$

where

$$\{n\} = \frac{1 - q^n}{1 - q} ,$$

is a so-called  $q$ -number and  $\{n\} \rightarrow n$ , if  $q \rightarrow 1$ . It is evident that if the parameter  $q$  is a real number the spectra of (17), (28) are real.

The algebra (39) has a realization in terms of  $q$ -derivative and the operator of multiplication (see, for example, [4])

$$a = \mathcal{D}_q, \quad b = y , \quad (41)$$

with the same  $q$  as in (39). This realization has a property that the vacuum remains the same as in the cases (18)-(19) and without loss of generality it can be set as  $|0\rangle = 1$ . Now we can substitute (41) in (17) and the following operator emerges

$$\begin{aligned} h_q^{(1)}(y, \mathcal{D}_q) = & -2\tilde{J}^0 \tilde{J}^- + 2\tilde{J}^0 - 2(p + \frac{1}{2})\tilde{J}^- = \\ & -2y\mathcal{D}_q^2 + 2(y - p - \frac{1}{2})\mathcal{D}_q , \end{aligned} \quad (42)$$

where the generators  $\tilde{J}^0 = ba$ ,  $\tilde{J}^- = a$  have the same functional form as in (10) but obey the  $q$ -deformed commutation relation

$$[\tilde{J}^0, \tilde{J}^-]_{1/q} \equiv \tilde{J}^0 \tilde{J}^- - \frac{1}{q} \tilde{J}^0 \tilde{J}^- = -\tilde{J}^- ,$$

forming the  $q$ -deformed Borel subalgebra  $b(2)_q$  of the  $q$ -deformed algebra  $sl(2)_q$  [9] (for discussion see [3]). In this case the operators (17), (28) are the  $sl(2)_q$ -*exactly-solvable operators*. Moreover, the operator (17) (as well as (28)) can be called the  *$q$ -deformed harmonic oscillator Hamiltonian in Fock space possessing  $sl(2)_q$  hidden algebra*.

The operator  $h_q(y, \mathcal{D}_q)$  is a non-local, three-point, discrete, dilatation-covariant operator defined on exponential lattice. It is illustrated by Fig.3.

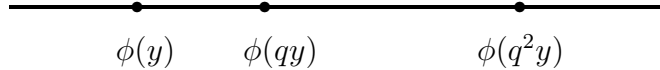


Fig. 3. Graphical representation of the operator (42)

The operator (42) can be called *the algebraic form of the Hamiltonian of the  $q$ -discretized harmonic oscillator*.

The spectral problem for the operator (42) has a form

$$-\frac{2}{yq(q-1)^2}\phi(q^2y) + \left[ \frac{2+q+q^2-2pq(1-q)}{q(q-1)^2} \frac{1}{y} + \frac{2}{1-q} \right] \phi(qy) - \left[ \frac{1+q-2p(1-q)}{y(q-1)^2} + \frac{2}{1-q} \right] \phi(y) = E^{(q)}\phi(y). \quad (43)$$

or, the r.h.s. can be taken as

$$= E^{(q)}\phi(qy). \quad (44)$$

or as

$$= E^{(q)}\phi(q^2y). \quad (45)$$

Usually, the spectral problem for  $q$ -discrete operators is defined with (44) as the r.h.s. (see, for example, [10]). It assumes that the middle point in Fig.3 remains fixed under dilatation. Introducing the new variable  $\tilde{y} = qy$ , it can be seen explicitly.

If in the case (43) the eigenvalues are given by (40) while for (44), (45) the eigenvalues are equal to

$$E_n^{(q)} = -2\{-n\}, \quad n = 0, 1, \dots \quad (46)$$

$$E_n^{(q)} = 2q^{-2n}\{n\}, \quad n = 0, 1, \dots \quad (47)$$

correspondingly. In the limit  $q \rightarrow 1$  all three expressions coincide corresponding to the original harmonic oscillator spectrum. The spectral problems (43)–(45) can be considered as a possible definition of a  $q$ -deformed harmonic oscillator. In the literature it is well-known many other definitions of the  $q$ -deformed harmonic oscillator (see for example, [11], [12] and references therein, [13])<sup>8</sup>. Such a situation reflects an ambiguity of making a  $q$ -deformation<sup>9</sup> as well as absence of clear physical criteria, which can remove or reduce this ambiguity. For instance, in the literature it is exploited three different types of the  $q$ -Laguerre polynomials (see, for example, an excellent review [14]), but it is not clear why other possible  $q$ -deformations of Laguerre polynomials are not studied.

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<sup>8</sup>Usually, these deformations are done by a direct discretization of the original Hamiltonian (1). Most of all are based on a discretization of the Infeld-Hall factorization representation of (1)

<sup>9</sup>For example, any term in non-deformed expression can be modified by multipliers of the type  $q^a$  and even some extra terms can be added with vanishing coefficients in the limit  $q \rightarrow 1$  like  $(1-q)^b$ ,  $b > 0$



Isospectrality of (17) and (28) is preserved by the  $q$ -deformation. Substitution of (41) in (28) gives a slight modification of the expressions (42). Unlike translation-covariant case it does not lead to a change of the nature of non-locality changing the number of points in the operator (42) as it is happened for the operators (20) and (29). The  $q$ -deformation of another algebraic form of the harmonic oscillator (34) in Fock space

$$h_q^{(2)}(b, a) = -\frac{1}{2\omega} \tilde{J}^- \tilde{J}^- + \tilde{J}^0, \quad (48)$$

(cf.(35)) takes in terms of the  $q$ -derivative the following form

$$h_q^{(2)}(x, \mathcal{D}_q) = -\frac{1}{2\omega} \mathcal{D}_q^2 + x \mathcal{D}_q, \quad (49)$$

(cf.(42)). It should be mentioned that the operators (42) and (49) are defined on the essentially different lattices, which are exponential in  $x$ - and  $y$ -variables, respectively.

Similar to what was done for canonical transformations it seems natural to introduce a notion of  $q$ -deformed canonical transformations, when two  $q$ -deformed systems are  $q$ -canonically equivalent if they can be connected through the  $q$ -deformed canonical transformation (see below). However, we were unable to find well-defined, non-trivial realization of the algebra (39) other than (40) which, for instance, would be similar to the realization (20) for (8) (see discussion in [4]).

#### 4. Anharmonic oscillator (perturbation theory)

Anharmonic oscillator is one of the most important non-exactly-solvable problems of quantum mechanics. One of the simplest and the most popular examples is given by the Hamiltonian

$$\mathcal{H}^{(aho)} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\omega^2}{2} x^2 + g x^{2n}, \quad n = 2, 3, 4, \dots \quad (50)$$

where  $g$  is a coupling constant<sup>10</sup>. Making first the gauge rotation of (50) with the gauge factor (4) we get

$$\begin{aligned} h^{(aho)}(y, \partial_y) &= \frac{1}{\omega} (\Psi_0^{(p)}(x))^{-1} \mathcal{H}^{(aho)} \Psi_0^{(p)}(x) \big|_{y=\omega x^2} \\ &= -2y \partial_y^2 + 2(y - p - \frac{1}{2}) \partial_y + \lambda y^n, \end{aligned} \quad (51)$$

(cf.(5)), where  $p = 0, 1$  and has a meaning of parity and  $\lambda = g\omega^{-(n+1)}$ . By the reasons which will be clear later the operator (51) can be called

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<sup>10</sup>So far, the most comprehensive study of the anharmonic oscillator (50) was carried out by Bender-Wu in the classical paper [15] (see also [16])

*algebraic form of the anharmonic oscillator Hamiltonian.* Replacing in (51) the derivative and the coordinate by the elements of the Heisenberg algebra (8),  $\partial_y \rightarrow a, y \rightarrow b$  we will arrive at the element of the Heisenberg-Weyl algebra

$$h^{(aho)}(b, a) = -2ba^2 + 2(b - p - \frac{1}{2})a + \lambda b^n, \quad (52)$$

which can be called the *anharmonic oscillator Hamiltonian in the Fock space*. Now we can fix  $n = 2$  and we will consider this particular case of the *quartic anharmonic oscillator* as the major example in further consideration.

Let us find a finite-difference operator which is canonically equivalent to (50). In order to do it we simply substitute the realization (19) of the Heisenberg algebra to the operator (52)

$$h_\delta^{(aho)}(y, D_\pm) = -\frac{2}{\delta}[y + \alpha + \delta(p + \frac{1}{2})]D_+ + 2[(1 + \frac{1}{\delta})(y + \alpha) - \frac{\delta\lambda}{2}(y + \alpha)^{(2)}]D_- + \frac{\delta^2\lambda}{2}(y + \alpha)^{(2)}D_-D_- + \frac{\lambda}{2}(y + \alpha)^{(2)}. \quad (53)$$

(see Fig.4). It is quite amazing that the perturbation  $\lambda b^2$  leads solely to an addition of one more point (marked by  $\diamond$ ) to the harmonic oscillator operator. A model characterized by the operator (53) can be called a *finite-difference anharmonic oscillator*.

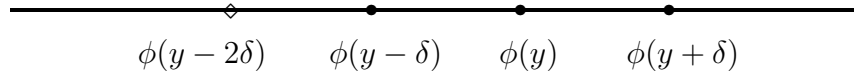


Fig. 4. Graphical representation of the problem (53)

It is non-local, four-point, finite-difference operator (for comparison see Fig.1 and Fig.2 with  $\delta \rightarrow -\delta$ ).

The corresponding spectral problem (\*) looks as follows

$$-\frac{2}{\delta^2}[y + \alpha + \delta(p + \frac{1}{2})]\phi(y + \delta) + \frac{2}{\delta}[(1 + \frac{2}{\delta})y + \alpha + p + \frac{1}{2}]\phi(y) - \frac{2}{\delta}(1 + \frac{1}{\delta})(y + \alpha)\phi(y - \delta) + \lambda(y + \alpha)^{(2)}\phi(y - 2\delta) = E\phi(y). \quad (54)$$

Their eigenvalues coincides with those of the anharmonic oscillator (50)–(52).

We have to emphasize that a presence of the anharmonic term changes the nature of non-locality of the harmonic oscillator leading to appearance an extra point. In particular case  $\delta = -1$  the number of points is reduced to three, however, the lattice becomes non-uniform.

In similar way one can construct a  $q$ -deformed anharmonic oscillator taking the operator (52) as an element of the  $q$ -Fock space. Substituting the realization (41) of the  $q$ -deformed Heisenberg algebra into (52) we get

$$h_q^{(aho)}(y, \mathcal{D}_q) = -2y\mathcal{D}_q^2 + 2(y - p - \frac{1}{2})\mathcal{D}_q + \lambda y^2, \quad (55)$$

(cf. (42)) which three-point, non-local operator (see, for example, Fig.3).

The spectral problem (\*) for the operator (55) has a form

$$\begin{aligned} & -2\frac{\phi(q^2y)}{yq(q-1)^2} + 2\frac{1+q+(y-p-\frac{1}{2})q(1-q)}{yq(q-1)^2}\phi(qy) - \\ & \frac{2-(2p+1)(1-q)+2y(1-q)-\lambda y^2(q-1)^2}{y(q-1)^2}\phi(y) = E^{(q)}\phi(y). \end{aligned} \quad (56)$$

(cf.(42)). Certainly, the r.h.s. in (56) can vary being equal to either (44), or (45). It reflects a fact that in the case of the  $q$ -Fock space the spectral problem (\*) can be modified in one way

$$L(b, a)\phi(b)|0 > = \lambda\phi(qb)|0 >, \quad (**)$$

or another

$$L(b, a)\phi(b)|0 > = \lambda\phi(q^2b)|0 >. \quad (***)$$

If we introduce the multiplication operator  $T_q f(x) = f(qx)$  the problems (\*\*) or (\*\*\*) correspond to an appearance of non-trivial operator weight factors in r.h.s.

$$L(b, a)\phi(b)|0 > = \lambda T_q \phi(b)|0 >, \quad (57)$$

or

$$L(b, a)\phi(b)|0 > = \lambda T_q^2 \phi(b)|0 >, \quad (58)$$

respectively.

It can be easily shown that the operator (52) describing the anharmonic oscillator does not belong to the class of the  $sl(2)$ -exactly-solvable operators. Hence their eigenfunctions are not polynomials. However, it was proved in [17] that in framework of some perturbative approach (see below) as a consequence of the fact that the perturbation  $b^2 \in \mathcal{P}_2(b)$  (see (12)) the perturbation theory in powers of the

parameter  $\lambda$  is algebraic one: any correction to any eigenfunction is a finite-order polynomial and hence can be found by algebraic means. In fact, such a perturbative approach provides a certain regular way to define an object in the Heisenberg-Weyl algebra which can be called the Hamiltonian of the quartic anharmonic oscillator. From practical point of view such a perturbation theory has very important feature: once being developed for the operator (52) it gives a unique possibility to construct *simultaneously (!)* the perturbation theory for the operators (50), (54), (56).

Let us approach to a construction of the above-mentioned perturbation theory. Following a standard prescription we take the spectral problem (\*) with

$$L(b, a) = h_0 + \lambda h_1$$

and develop a perturbation theory in powers of  $\lambda$  searching for corrections in a form

$$\phi = \sum \lambda^n \phi_n, \quad E = \sum \lambda^n E_n. \quad (57)$$

Collecting the terms of the order  $\lambda^n$  it is easy to derive an equation for the  $n$ th correction

$$(h_0 - E_0)\phi_n = \sum_{i=1}^n E_i \phi_{n-i} - h_1 \phi_{n-1}. \quad (58)$$

A remarkable feature of this form of perturbation theory a possibility to study a single state separately, without touching other states as it was the case of the Rayleigh-Schroedinger form of perturbation theory.

Now we take as the unperturbed operator  $h_0$  the Hamiltonian of the harmonic oscillator in the Fock space (17) and consider as the perturbation  $h_1 = b^2$ . As it was mentioned already, in this case the  $n$ th correction  $\phi_n$  to eigenfunction should be polynomials in  $b$ . As an instructive example let us study the ground state of anharmonic oscillator. The ground state of the unperturbed problem (17) is characterized by

$$\phi_0 = 1, \quad E_0 = 1. \quad (59)$$

By solving the equation (58) after simple calculations we can get the explicit form for the first several corrections, for example,

$$-\phi_1 = \frac{b^2}{2\{2\}} + \frac{(3+2p)}{4}b, \quad E_1 = \frac{(1+2p)(3+2p)}{4},$$

and

$$\phi_2 = \frac{1}{4} \left[ \frac{1}{\{2\}\{4\}} b^4 + \frac{2q^2 + 5q + 6 + 2p(q+2)}{2\{2\}\{3\}} b^3 + \right.$$

$$\frac{4q^2 + 12q + 15 + 8p(q + 2) + 4p^2}{4\{2\}}b^2 + \frac{(3 + 2p)[q^2 + 3q + 3 + 2p(q + 1)]}{2}b \Big],$$

$$E_2 = -\frac{(1 + 2p)(3 + 2p)}{8}[q^2 + 3q + 3 + 2p(q + 1)] . \quad (60)$$

Without any difficulties one can find several next corrections. However, it becomes evident very quickly that the complexity of calculations is growing very fast with a number of correction. Using different realizations of the  $(q)$ -Heisenberg algebra one can calculate perturbative corrections to the various forms of anharmonically-perturbed harmonic oscillator (differential, finite-difference, discrete).

I.  $q = 1, b = y$ .

This case corresponds to the coordinate-momentum realization of the Heisenberg algebra (18) and vacuum definition  $|0\rangle = 1$ . It leads to a standard anharmonic oscillator (50) and the corrections are:

$$-\phi_1 = \frac{y^2}{4} + \frac{(3 + 2p)}{4}y ,$$

and

$$\phi_2 = \frac{1}{4} \left[ \frac{y^4}{8} + \frac{11 + 6p}{12}y^3 + \frac{31 + 24p + 4p^2}{8}y^2 + \frac{(3 + 2p)(7 + 4p)}{2}y \right] , \quad E_2 = -\frac{(1 + 2p)(3 + 2p)(7 + 4p)}{8} , \quad (61)$$

where  $E_1$  is the same as in (60).

Given form of the perturbation theory coincides to the so-called ‘ $F$ -functions method’ developed by Dalgarno (see discussion in [16]), which in fact was realized for the case of the anharmonic oscillator (50) in [15]. It is easy to check that the corrections (61) coincide to those calculated in text-books (see for example [5]).

II.  $q = 1, b = (y + \alpha)(1 - \delta\mathcal{D}_-)$  (see (19))

This case corresponds to the translation-covariant discretization and perturbed finite-difference harmonic oscillator (20) (see (53)):

$$-\phi_1 = \frac{\tilde{y}^{(2)}}{4} + \frac{(3 + 2p)}{4}\tilde{y} ,$$

and

$$\phi_2 = \frac{1}{4} \left[ \frac{\tilde{y}^{(4)}}{8} + \frac{11+6p}{12} \tilde{y}^{(3)} + \frac{31+24p+4p^2}{8} \tilde{y}^{(2)} + \frac{(3+2p)(7+4p)}{2} \tilde{y} \right], \quad (62)$$

where  $E_1$  is the same as in (60) and  $E_2$  is the same as in (61). Here  $\tilde{y}^{(n+1)} = (y + \alpha)^{(n+1)} = (y + \alpha)(y + \alpha - \delta) \dots (y + \alpha - n\delta)$  is quasi-monomial.

III.  $q \neq 1$ ,  $b = y$ .

This case corresponds to the perturbed  $q$ -harmonic oscillator (55)-(56) and corrections are equal to

$$-\phi_1 = \frac{y^2}{2\{2\}} + \frac{(3+2p)}{4}y, \quad E_1 = \frac{(1+2p)(3+2p)}{4}$$

and

$$\begin{aligned} \phi_2 = & \frac{1}{4} \left[ \frac{1}{\{2\}\{4\}} y^4 + \frac{2q^2 + 5q + 6 + 2p(q+2)}{2\{2\}\{3\}} y^3 + \right. \\ & \left. \frac{4q^2 + 12q + 15 + 8p(q+2) + 4p^2}{4\{2\}} y^2 + \frac{(3+2p)[q^2 + 3q + 3 + 2p(q+1)]}{2} y \right], \\ E_2 = & -\frac{(1+2p)(3+2p)}{8} [q^2 + 3q + 3 + 2p(q+1)]. \end{aligned} \quad (63)$$

It is quite interesting to see how above-mentioned results will be modified if instead of the spectral problem (\*) the spectral problem (\*\*) is considered. The equation (58) for the  $n$ th correction becomes

$$(h_0 - E_0 T_q) \phi_n = \sum_{i=1}^n E_i T_q \phi_{n-i} - h_1 \phi_{n-1}, \quad (64)$$

(cf. (58)). The ground state of the unperturbed problem (59) is unchanged. The first correction is also unchanged while the second correction now takes a modified form

$$\begin{aligned} \phi_2 = & \frac{1}{4} \left[ \frac{1}{\{2\}\{4\}} b^4 + \frac{2q^2 + 5q + 6 + 2p(q+2)}{2\{2\}\{3\}} b^3 + \right. \\ & \left. \frac{4q^2 + 12q + 15 + 8p(q+2) + 4p^2}{4\{2\}} b^2 + \frac{(3+2p)[q^2 + 3q + 3 + 2p(q+1)]}{2} b \right], \end{aligned}$$

$$E_2 = -\frac{(1+2p)(3+2p)}{64} [65 - 9q^2 + 8p(7-3q^2) + 12p^2(1-q^2)] . \quad (65)$$

### 5. Anharmonic oscillator (quasi-exactly-solvable model)

Among one-dimensional Schroedinger equations there is some class of problems possessing a certain outstanding property - first several eigenstates can be found explicitly, by algebraic means. Such problems are called *quasi-exactly-solvable* [18]. Among ten known families of one-dimensional quasi-exactly-solvable potentials [19] there exists one, which can be treated as an anharmonic oscillator and its Hamiltonian is

$$\mathcal{H}^{(qes)} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \left[ \frac{\omega^2}{2} - (2n + \frac{3}{2} + p)g \right] x^2 + g\omega x^4 + \frac{g^2}{2} x^6 . \quad (66)$$

Here  $g$  is a coupling constant and  $x \in (-\infty, \infty)$ . Parameter  $p = 0, 1$  has meaning of parity. The first  $(n+1)$  eigenfunctions of parity  $p$  (but not others) are of the form

$$\Psi_n^{(p)}(x) = x^p P_n(\omega x^2) e^{-\omega x^2/2 - gx^4/4} \quad (67)$$

where  $P_n(y)$  is a polynomial of degree  $n$ .

Making first the gauge rotation of (50) with the gauge factor (67) at  $n = 0$  we get

$$\begin{aligned} h^{(qes)}(y, \partial_y) &= \frac{1}{\omega} (\Psi_0^{(p)}(x))^{-1} \mathcal{H}^{(qes)} \Psi_0^{(p)}(x) |_{y=\omega x^2} \\ &= -2y \partial_y^2 + 2(\lambda y^2 + y - p - \frac{1}{2}) \partial_y - 2\lambda n y , \end{aligned} \quad (68)$$

(cf. (5),(51)), where  $\lambda = g\omega^{-2}$  and constant terms are dropped out. The spectral problem for (68) is defined on the half-line,  $y \in [0, \infty)$ . The first  $(n+1)$  eigenfunctions of (68) are some polynomials of the degree  $n$ ,  $P_n(y)$  (cf.(67)) possessing  $k = 0, 1, \dots, n$  real zeroes inside of the interval  $y \in [0, \infty)$ .

Replacing in (68) the derivative and the coordinate by the elements of the Heisenberg algebra (8):  $\partial_y \rightarrow a$ ,  $y \rightarrow b$  we arrive at the element of the Heisenberg-Weyl algebra

$$h^{(qes)}(b, a) = -2ba^2 + 2(\lambda b^2 + b - p - \frac{1}{2})a - 2\lambda n b , \quad (69)$$

(cf. (17),(28),(52)). As in previous consideration the operator  $h^{(qes)}(b, a)$  can be treated as an element of the Fock space as well as an element of

the  $q$ -Fock space. This operator can be called the *Fock space Hamiltonian of the anharmonic quasi-exactly-solvable oscillator*. The operator  $h^{(qes)}(b, a)$  in the Fock space is  $sl_2$ -quasi-exactly-solvable operator. It can be rewritten in terms of the generators of  $sl_2$ -algebra (10)

$$h^{(qes)}(b, a) = -2J_n^0 J_n^- + 2\lambda J_n^+ + 2J_n^0 - (n + 1 + 2p)J_n^- , \quad (70)$$

where as always the constant terms are dropped off.

However, it is easy to check that in the  $q$ -Fock space the operator  $h^{(qes)}(b, a)$  is *not*  $sl_{2q}$ -quasi-exactly-solvable operator, since it can not be rewritten in terms of the generators of  $sl_{2q}$ -algebra<sup>11</sup>. This operator needs a slight modification of the last term in order to become the  $sl_{2q}$ -quasi-exactly-solvable:

$$h^{(qes)}(b, a) = -2ba^2 + 2(\lambda b^2 + b - p - \frac{1}{2})a - 2\lambda\{n\}b = \quad (71)$$

$$-2\tilde{J}_n^0 \tilde{J}_n^- + 2\lambda\tilde{J}_n^+ + 2\tilde{J}_n^0 - \left(2\frac{\{n\}\{n+1\}}{\{2n+2\}} + 1 + 2p\right)\tilde{J}_n^- ,$$

(cf. (69)), where  $\{n\}$  is the  $q$ -number (see (31)) and the constant terms in the second expression are dropped off.

The first  $(n + 1)$  eigenfunctions of (71) are some polynomials in  $b$  of degree  $n$ ,  $P_n(b)$ . If in the case  $n = 0$  the first eigenfunctions for the spectral problems (\*) and (\*\*) coincide and equal to:

$$\phi^{(0)} = 1 , \quad E^{(0)} = 0 ,$$

while for the case of  $n = 1$  they are already different. Namely, for the spectral problem (\*),

$$\begin{aligned} \phi_{\pm}^{(1)} &= b - \frac{1 \mp \sqrt{1 + 4\lambda(1 + 2p)}}{4\lambda} , \\ E_{\pm}^{(1)} &= \frac{1 \mp \sqrt{1 + 4\lambda(1 + 2p)}}{2} , \end{aligned} \quad (72)$$

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<sup>11</sup>In the case of the  $sl_{2q}$ -algebra the major modification of (10) comes for the positive-root generator

$$J_n^+ = b^2 a - nb \rightarrow \tilde{J}_n^+ = b^2 a - \{n\}b ,$$

while the Cartan generator

$$J_n^0 = ba - \frac{n}{2} \rightarrow \tilde{J}_n^0 = ba - \frac{\{n\}\{n+1\}}{\{2n+2\}} ,$$

where  $\{n\}$  is the  $q$ -number (see (31)) and  $J^-$  remains unchanged; for details, see [4]



where sub-indices (+) and (−) are assigned to the ground and the first excited state, respectively. As for the spectral problem (\*\*) the formulas (72) are modified

$$\begin{aligned}\phi_{\pm}^{(1)} &= b - \frac{1 \mp \sqrt{1 + 4\lambda q(1 + 2p)}}{4\lambda}, \\ E_{\pm}^{(1)} &= \frac{1 \mp \sqrt{1 + 4\lambda q(1 + 2p)}}{2q}.\end{aligned}\quad (73)$$

Of course, when  $q = 1$  the formulas (72) and (73) coincide.

Now we can find a finite-difference operator canonically equivalent to (68). In order to do it we put  $q = 1$  in (68) and substitute the realization (19) of the Heisenberg algebra into the operator (69)

$$\begin{aligned}h_{\delta}^{(qes)}(y, D_{\pm}) &= \\ -\frac{2}{\delta}[y + \alpha + \delta(p + \frac{1}{2})]D_+ + 2[(1 + \frac{1}{\delta})(y + \alpha) + \lambda(y + \alpha)(y + \alpha + \delta(n - 1))]D_- \\ - 2\lambda\delta(y + \alpha)^{(2)}D_-D_- - 2\lambda n(y + \alpha),\end{aligned}\quad (74)$$

(cf. (53)). It turns out that this operator is four-point finite-difference operator (see Fig.4). It is quite amazing that independently on  $n$  the perturbation  $2\lambda b(ba - n)$  to the harmonic oscillator operator (see Fig.1) leads solely to an appearance of one extra point (marked by  $\diamond$  in Fig.4) similarly to (53). A model characterized by the operator (74) can be called a *finite-difference quasi-exactly-solvable anharmonic oscillator*.

Without loss of generality we place  $\alpha = 0$  in (74) and the corresponding spectral problem (\*) for (74) looks like

$$\begin{aligned}-\frac{2}{\delta}\left[\frac{y}{\delta} + p + \frac{1}{2}\right]\phi(y + \delta) + \frac{2}{\delta}\left[(1 + \frac{2}{\delta})y + p + \frac{1}{2}\right]\phi(y) \\ - \frac{2}{\delta}y\left[(1 + \frac{1}{\delta}) - \lambda(y - \delta(n + 1))\right]\phi(y - \delta) \\ - 2\lambda\frac{y^{(2)}}{\delta}\phi(y - 2\delta) = E\phi(y).\end{aligned}\quad (75)$$

It is natural to assume that the spectral problem (75) is defined in  $y \in [0, \infty)$ . Their first  $(n + 1)$  eigenvalues coincide with those of the anharmonic oscillator Hamiltonian (66) as well as the operators (68), (69). Their eigenfunctions remain polynomials but modified – each monomial should be replaced by quasi-monomial,  $y^k \rightarrow y^{(k)}$ . For instance, the formulas (72)–(73) become

$$\phi_{\pm}^{(1)} = y - \frac{1 \mp \sqrt{1 + 4\lambda(1 + 2p)}}{4\lambda},$$

$$E_{\pm}^{(1)} = \frac{1 \mp \sqrt{1 + 4\lambda(1 + 2p)}}{2} . \quad (76)$$

It is worth to note that for physically relevant parameters  $g \geq 0$  and  $q \geq 0$ <sup>12</sup> the eigenfunctions (72), (76) in domain  $y \geq 0$  have no nodes for ground state and have the only one node for the first excited state like an analogue of the oscillation (Sturm) theorem holds (see, for example, [5]). Likely, an agreement with Sturm theorem will hold for higher excited states but we were unable to prove it in full generality.

## 6. CONCLUSION

We introduced a simple-minded notion of canonical equivalence in quantum mechanics. Then we showed that for canonically-equivalent systems the spectra remain the same and constructed discrete systems, which are canonically-equivalent to the harmonic and a certain type of anharmonic oscillators – the quartic oscillator and the quasi-exactly-solvable oscillator. In general, these discrete systems are defined on non-linear lattices.

We restricted our studies to the operators in the algebraic form possessing the polynomial eigenfunctions. An important question to ask is what would happen if the operator in hand possesses an algebraic form but non-polynomial eigenfunctions. Perhaps, the most explicit instructive example is given by the original Hamiltonian (1) of the harmonic oscillator. Following our philosophy it can be rewritten as the element of the Fock space

$$\mathcal{H} = -\frac{1}{2}a^2 + \frac{\omega^2}{2}b^2 , \quad (77)$$

where its ground state eigenfunction can be written formally as

$$\Psi_0(b)|0\rangle = \sum_{n=0}^{\infty} \frac{(-\omega)^n}{2^n n!} b^{2n} |0\rangle . \quad (78)$$

Substitution of (19) with  $|0\rangle = 1$  into (78) leads to an infinite series of the form

$$\tilde{\Psi}_0(x) = \sum_{n=0}^{\infty} \frac{(-\omega)^n}{2^n n!} x^{(2n)} , \quad (79)$$

which has zero radius of convergence in  $x$  for any  $\delta \neq 0$ <sup>13</sup>. Although the Taylor expansion in powers of  $x$  for the original eigenfunction (4)

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<sup>12</sup>In this case  $q$  has the meaning the parameter of dilatation

<sup>13</sup>This type of expansion on the basic set of quasi-monomials is known in literature as the Newton series. For discussion see, for example, [7]

at  $p = 0$  had infinite radius of convergence. So, the radius of convergence has delta-function behaviour. Perhaps, one of possible ways to remedy this drawback is to gauge-rotate a Hamiltonian with non-polynomial eigenfunctions using a semi-classical or somehow modified semi-classical wavefunction as a gauge factor requiring an appearance of an algebraic form of the final operator.

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